INVERSE COEFFICIENT PROBLEM FOR A DEGENERATE PARABOLIC EQUATION WITH THE BESSEL OPERATOR

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Abstract
We study a direct and an inverse coefficient problem for a degenerate parabolic equation involving the Euler-Bessel operator with a general degeneracy parameter $v \in [0,1)$. The unknown time-dependent coefficient $r(t)$ is recovered from an integral overdetermination condition. Using the Fourier-Bessel spectral method, the inverse problem is reduced to a Volterra integral equation of the second kind, for which existence and uniqueness of the solution pair are established. A Lipschitz stability estimate is obtained, showing that the inverse problem is conditionally well posed.

Keywords: *degenerate parabolic equation; Euler-Bessel operator; inverse coe-cient problem; integral overdetermination condition; Volterra integral equation; Lipschitz stability.*

1 INTRODUCTION

A central problem in the theory of parabolic equations is the recovery of an unknown coefficient from indirect measurements of the solution the so-called inverse coefficient problem. Such problems are generally ill posed in the sense of Hadamard, and their rigorous treatment requires proving both existence and uniqueness of the solution pair. A natural and effective approach is to supplement the governing equation with an integral overdetermination condition, which allows the unknown coefficient to be characterized as the solution of a Volterra integral equation of the second kind [4, 7].

Inverse problems for degenerate parabolic equations have attracted considerable attention [1, 2, 3]. The presence of the Euler–Bessel operator $x^{-v}\partial_x(x^v\cdot)$ introduces a degeneracy at $x=0$ whose strength is governed by the parameter v; the associated spectral theory is built upon Bessel functions [6]. Recently, inverse coefficient problems for diffusion equations with a Bessel operator under integral overdetermination were studied in [5].

In the present work we treat the recovery of a time-dependent coefficient $r(t)$ for the classical degenerate equation with the Euler–Bessel operator, establishing existence, uniqueness, and conditional stability.

2. Statement of the problem

In the rectangle $\Omega=(0,l)\times(0,T)$ we consider the degenerate parabolic equation

$$u_t = x^{-v}\frac{\partial}{\partial x}\left(x^v u_x\right) + r(t) f(x,t), \qquad (x,t)\in\Omega, \quad 0\le v<1, \tag{1}$$

with the following initial and boundary conditions

$$u(x,0)=\varphi(x), \quad x\in[0,l], \quad u(0,t)=u(l,t)=0, \quad t\in[0,T], \tag{2}$$

where $\varphi(x), f(x,t)$ are given functions such that $\varphi(0)=\varphi(l)=0$.

Direct problem (DP1). For $r(t)=1$, find $u(x,t)$ satisfying (1)-(2) and the regularity conditions $u(x,t), x^{v}u_{x}(x,t)\in C(\overline{\Omega})$ and $x^{-v}\frac{\partial}{\partial x}(x^{v}u_{x}), u_{t}\in C(\Omega)$.

Inverse problem (IP1). Find a pair $\{u(x,t), r(t)\}$ with $r(t)\in C([0,T])$ such that $u(x,t)$ solving (1)-(2) and the integral overdetermination condition

$$\int_0^l u(x,t)x^{v}w(x)dx=g(t),\ t\in[0,T], \tag{3}$$

hold, where $w(x)$ and $g(t)$ are given functions.

3. The spectral problem and the direct problem

Separation of variables for (1)-(2) leads to the spectral problem

$$x^{-v}\frac{\partial}{\partial x}(x^{v}X'(x))=-\lambda^{2}X(x),\ \ X(0)=X(l)=0. \tag{4}$$

With the substitution $X(x)=x^{(1-v)/2}Z(\lambda x)$, $p=(1-v)/2$, equation (4) reduces to the Bessel equation, so its eigenfunctions are

$$X_n(x)=x^{(1-v)/2}J_p(\lambda_n x),\qquad \lambda_n=\frac{\mu_n}{l},\qquad n=1,2,...,$$

where J_p is the Bessel function of the first kind and μ_n is the n-th positive zero of J_p. After normalization in $L_v^2(0,l)$ (weight $\rho(x)=x^{v}$) one obtains a complete orthonormal system $\{X_n(x)\}$. By the asymptotics of the Bessel zeros,

$$\lambda_n l=\pi n-\frac{v\pi}{4}+O\left(\frac{1}{n}\right),\quad \text{hence}\quad \lambda_n{}^{2}\le Cn^{2}. \tag{5}$$

Throughout we impose the compatibility conditions

$$\varphi, f(\cdot,t)\in C^{4}[0,l],\quad \varphi^{(j)}(0)=\varphi^{(j)}(l)=f^{(j)}(0,t)=f^{(j)}(l,t)=0,\ \ j=0,1,2. \tag{6}$$

Denote the Fourier–Bessel coefficients $\varphi_n=\int_0^l\varphi(x)x^{v}X_n(x)dx$ and $f_n(t)=\int_0^l f(x,t)x^{v}X_n(x)dx.$

Lemma 1. Under conditions (6) there exist constants $C_1,C_2>0$ such that

$$|\varphi_n|\le\frac{C_1}{n^{4}},\qquad |f_n(t)|\le\frac{C_2}{n^{4}},\qquad n=1,2,...,$$

The decay rate in Lemma 1, obtained by repeated integration by parts, guarantees the uniform convergence of the relevant series. For $r(t)=1$ the solution of (DP1) admits the Fourier–Bessel representation

$$u(x,t)=\sum_{n=1}^{+\infty}T_n(t)X_n(x),\ \ T_n(t)=\varphi_n e^{-\lambda_n^{2}t}+\int_0^t f_n(\tau)e^{-\lambda_n^{2}(t-\tau)}d\tau. \tag{7}$$

Theorem 1. Let $\varphi(x)$ and f satisfy conditions (6). Then the direct problem (DP1) has a unique solution, given by (7).

4. The inverse problem

For a general coefficient $r(t)\in C([0,T])$, the solution representation (7) extends to

$$u(x,t)=\sum_{n=1}^{+\infty}\varphi_n e^{-\lambda_n^2 t}X_n(x)+\sum_{n=1}^{+\infty}\left(\int_0^t r(\tau)f_n(\tau)e^{-\lambda_n^2(t-\tau)}d\tau\right)X_n(x). \tag{8}$$

Substituting (8) into the overdetermination condition (3) and differentiating in t (the series converge uniformly in $\overline{\Omega}$, so term-wise differentiation is justified), we obtain, after solving for $r(t)$, the Volterra integral equation of the second kind

$$r(t)=A(t)+\int_0^t K(t,\tau)r(\tau)d\tau, \tag{9}$$

with

$$A(t)=\frac{g'(t)+\sum_{n=1}^{+\infty}\lambda_n^2\varphi_n e^{-\lambda_n^2 t}\int_0^l x^v w(x)X_n(x)dx}{\int_0^l x^v w(x)f(x,t)dx},$$

$$K(t,\tau)=\frac{\sum_{n=1}^{+\infty}\lambda_n^2 f_n(\tau)e^{-\lambda_n^2(t-\tau)}\int_0^l x^v w(x)X_n(x)dx}{\int_0^l x^v w(x)f(x,t)dx}.$$

Using Lemma 1, the bound $\lambda_n^2\le Cn^2$ from (5), the regularity $w(x)\in C^4[0,l]$, and the Cauchy–Schwarz inequality, the series defining $A(t)$ and $K(t,\tau)$ converge absolutely and define continuous functions. Hence (9) is a Volterra equation of the second kind with continuous kernel, which possesses a unique continuous solution.

Theorem 2 (existence and uniqueness for IP1). Assume that $\varphi(x)$ and f satisfy (6); $w(x)\in C^4[0,l]$ with $w(0)=w(l)=0$; $g(t)\in C^1([0,T])$; and

$$\int_0^l x^v w(x)f(x,t)dx\neq 0 \qquad \text{for all } t\in[0,T].$$

Then the inverse problem {(1),(2),(3)} admits a unique solution pair $\{r(t),u(x,t)\}$ with $r(t)\in C([0,T])$ and $u(x,t),x^v u_x(x,t)\in C(\overline{\Omega})$, $x^{-v}\frac{\partial}{\partial x}(x^v u_x),u_t\in C(\Omega)$.

The stability of the recovered coefficient follows from the same Volterra formulation.

Theorem 3 (Lipschitz stability of IP1). Let the hypotheses of Theorem 2 hold, and let $\{r_1,u_1\}$, $\{r_2,u_2\}$ be the solution pairs corresponding to data $g_1,g_2\in C_1([0,T])$ with the same φ,f,w. Set

$$d_0=\min_{t\in[0,T]}\left|\int_0^l x^v w(x)f(x,t)dx\right|>0.$$

Then there exists a constant $C_{st}>0$, independent of g_1,g_2, such that

$$\|r_1-r_2\|_{C([0,T])}\le C_{st}\|g_1-g_2\|_{C^1([0,T])}. \tag{10}$$

Estimate (10) shows that the recovered coefficient depends continuously on the observation, the data being measured in the C^1-norm. Consequently, together with Theorem 2, the inverse problem (IP1) is conditionally well posed within its admissible class. We emphasize that the stability involves the derivative $\Delta g' = g_1' - g_2'$ of the data; in the weaker observation norm $\|g\|_{C([0,T])}$ alone the problem remains ill posed, the coefficient being controlled only after one derivative of the data is taken into account.

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G‘ulomjanova Sevaraxon G‘ofurjon qizi

19 THE INVERSE-SOURCE PROBLEM FOR MIXED-TYPE EQUATIONS
Murolimova N.A. **99**

20 INVERSE COEFFICIENT PROBLEM FOR A DEGENERATE PARABOLIC EQUATION WITH THE BESSEL OPERATOR
Kodiraliev Asadbek **102**

21 A BOUNDARY-VALUE PROBLEM FOR A COUPLED SECOND KIND DEGENERATE HYPERBOLIC EQUATIONS
Feruza Mamanazarova **106**

22 YUPQA PARDALAR TENZOQARSHILIGIGA TA'SIR ETUVCHI OMILLAR TAHLILI
To‘xtasinov Oybekjon Abduvahob o‘g‘li **110**

23 YARIMO‘TKAZGICHLAR VA IOT (INTERNET OF THINGS): RAQAMLI DUNYO TARMOG‘INING RIVOJI
To‘xtasinov Oybekjon Abduvahob o‘g‘li **113**

24 СВОЙСТВА СЛОЕВ НА ОСНОВЕ ОКСИДА МОЛИБДЕНА ДЛЯ ПОЛУПРОВОДНИКОВЫХ ГАЗОВЫХ ДАТЧИКОВ
Д.М.Ташланова
А.И.Зокиров **115**

25 FARG‘ONA VODIYSI IQLIM SHAROITIDA QUYOSH PANELLARI ISHLASH SAMARADORLIGIGA TA'SIR ETUVCHI ASOSIY OMILLARNING KOMPLEKS TAHLILI
Axmedova Shalola Yunusali qizi
Kamolova Muhabbatxon Muhiddin qizi
Holmatova Ruhshona **119**

26 TOR TEBRANISH TENGLAMASI UCHUN INTEGRAL SHARTLI MASALA
Jo‘rayeva Dilnavoz Umidjon qizi **126**

27 BESSEL OPERATORI QATNASHGAN ITERATSIYALANGAN ISSIQLIK TENGLAMALARINI TADQIQ ETISH
Raximova Sh.U. **129**

28 KUCHSIZ-SINGULYAR INTEGRAL OPERATORLAR UCHUN SHARTLANGAN ITERATSION METODLARNI TADQIQ QILISH
Raximov Quvvatali Ortiqovich **131**