EXISTENCE AND UNIQUENESS OF A SOLUTION TO A BITSADZE-SAMARSKII PROBLEM FOR A SUB-DIFFUSION EQUATION WITH THE REGULARIZED PRABHAKAR FRACTIONAL DERIVATIVE

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Abstract
This paper investigates the unique solvability of a Bitsadze-Samarskii problem for a sub-diffusion equation involving the regularized Prabhakar fractional derivative. By deriving the solution of the corresponding first boundary value problem, the nonlocal problem is reduced to a Volterra integral equation of the second kind. The continuity of the kernel and the right-hand side ensures the existence of a unique solution to the integral equation. Consequently, the solution of the Bitsadze-Samarskii problem is represented through the solutions of the associated Volterra integral equation and the first boundary value problem. An existence and uniqueness theorem is established.

Keywords: *Bitsadze-Samarskii problem, regularized Prabhakar fractional derivative, Prabhakar function, Volterra integral equation of the second kind.*

Let Ω be a rectangular domain: $\Omega=\{(t,x): 0<t<T,\ 0<x<a\},\ \ 0<a,T<+\infty.$ We formulate the Bitsadze-Samarskii problem for the following sub-diffusion equation:

$$ {}^{PC}D_{0t}^{\alpha,\beta,\gamma,\delta}u(t,x)-u_{xx}(t,x)=f(t,x). \tag{1}$$

Here $f(t,x)$ is a given function, α, β, γ, δ are given real numbers such that $\alpha>0$, $0<\beta\le 1$,

$$ {}^{PC}D_{0t}^{\alpha,\beta,\gamma,\delta}g(t)={}^{P}I_{0t}^{\alpha,1-\beta,-\gamma,\delta}\frac{d}{dt}g(t),\ \ t>0$$

is the regularized Prabhakar fractional derivative [1],

$$ {}^{P}I_{0t}^{\alpha,\beta,\gamma,\delta}g(t)=\int_0^t (t-s)^{\beta-1}E_{\alpha,\beta}^{\gamma}\left[\delta(t-s)^{\alpha}\right]g(s)ds,$$

is the Prabhakar fractional integral operator, whereas

$$ E_{\alpha,\beta}^{\gamma}[z]=\sum_{k=0}^{+\infty}\frac{(\gamma)_k z^k}{\Gamma(\alpha k+\beta)k!}$$

represents the Prabhakar function [2].

Definition. *A regular solution of the equation (1) in the domain Ω is called a function $u(t,x)$ with the regularity*

$$ u(t,x)\in C(\overline{\Omega}),\ \ u_{xx}(t,x)\ \ {}^{PC}D_{0t}^{\alpha,\beta,\gamma,\delta}u(t,x)\in C(\Omega)$$

that satisfies the equation (1) at all points $(t,x)\in\Omega$.

Bitsadze-Samarskii Problem. Find a regular solution $u(t,x)$ of the equation (1) in the domain Ω, satisfying the following initial and boundary conditions:

$$ u(0,x)=\tau(x),\quad 0\le x\le a, \tag{2}$$

$$u(t,0)=\varphi_0(t),\quad 0\le t\le T, \tag{3}$$

$$u(t,a)=b(t)u(t,\xi_0)+c(t),\quad 0\le t\le T, \tag{4}$$

where $\xi_0=const\in(0,a)$, $\tau(x)$, $\varphi_0(t)$, $b(t)$, $c(t)$ are the given functions.

If $b(t)\equiv 0$, the considered problem reduces to the first boundary value problem for the equation (1) in the domain Ω. Therefore, in what follows we assume that $b(t)\not\equiv 0$.

Let us assume that the solution $u(t,x)$ of the Bitsadze-Samarskii problem exists. If we introduce the notation

$$u(t,a)=\varphi_1(t),\quad t\in[0,T] \tag{5}$$

for the equation (1), the conditions (2), (3), (5) define the first boundary value problem. In [3], the following theorem was formulated for this first boundary value problem:

Theorem 1. *Let* $\varphi_0(t)$, $\varphi_1(t)\in C[0;T]$, $\tau(x)\in C[0;a]$, $f(t,x)=t^{-\varepsilon}f_1(t,x)$, $f_1(t,x)\in C(\bar{\Omega})$, $0\le\varepsilon<\beta_1$ *and the conditions*

$$\varphi_0(0)=\tau(0),\varphi_1(0)=\tau(a).$$

admits a unique regular solution given by

$$u(t,x)=\int_0^t\varphi_0(\eta)G_\xi(t,x,\eta,0)d\eta-\int_0^t\varphi_1(\eta)G_\xi(t,x,\eta,a)d\eta+$$
$$+\int_0^a\tau(\xi)G(t,x,0,\xi)d\xi+\int_0^t\int_0^a f(\eta,\xi)G(t,x,\eta,\xi)d\xi d\eta. \tag{6}$$

Here

$$G(t,x,\eta,\xi)=\frac{(t-\eta)^{\beta_1-1}}{2}\sum_{n=-\infty}^{\infty}\left[E_{12}\left(\begin{matrix}-\gamma_1,1,\gamma_1;\\ -\beta_1,\alpha,\beta_1;-\gamma_1,\gamma_1;1,1;1,1\end{matrix}\middle|\begin{matrix}-|x-\xi+2an|(t-\eta)^{-\beta_1}\\ \delta(t-\eta)^{\alpha}\end{matrix}\right)-\right.$$
$$\left.-E_{12}\left(\begin{matrix}-\gamma_1,1,\gamma_1;\\ -\beta_1,\alpha,\beta_1;-\gamma_1,\gamma_1;1,1;1,1\end{matrix}\middle|\begin{matrix}-|x+\xi+2an|(t-\eta)^{-\beta_1}\\ \delta(t-\eta)^{\alpha}\end{matrix}\right)\right],$$

$$G(t,x,0,\xi)=\int_0^t\eta^{-\beta}E^{-\gamma}_{\alpha,1-\beta}[\delta\eta^{\alpha}]G(t,x,\eta,\xi)d\eta.$$

$$E_{12}\left(\begin{matrix}\alpha_1,\lambda_1,\delta_1;\\ \alpha_2,\lambda_2,\delta_2;\alpha_3,\delta_3;\alpha_4,\delta_4;\lambda_3,\delta_5\end{matrix}\middle|\begin{matrix}x\\ y\end{matrix}\right)=$$
$$=\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty}\frac{\Gamma(\alpha_1n+\lambda_1m+\delta_1)x^ny^m}{\Gamma(\alpha_2n+\lambda_2m+\delta_2)\Gamma(\alpha_3n+\delta_3)\Gamma(\alpha_4n+\delta_4)\Gamma(\lambda_3m+\delta_5)},$$

$(x,y,\alpha_l,\lambda_i,\delta_j\in\mathbb{C};\min\{\alpha_l,\lambda_i\}>0;(l=\{1,...,4\},i=\{1,2,3\},j=\{1,...,5\}))$, in which the double series converges for $x,y\in\mathbb{C}$, if $\Delta_1>0$, and $\Delta_2>0$, whereas $\Delta_1=\alpha_2+\alpha_3+\alpha_4-\alpha_1$, $\Delta_2=\lambda_2+\lambda_3-\lambda_1$, $\beta_1=\frac{\beta}{2}$, $\gamma_1=\frac{\gamma}{2}$ [4].

For the considered Bitsadze-Samarskii problem, we formulate the following Lemma and Theorem.

Lemma 1. *Let* $\tau(x)\in C[0,a]$, $\varphi_0(t)$, $b(t)$, $c(t)\in C[0,T]$, $f(t,x)=t^{-\varepsilon}f_1(t,x)$, $f_1(t,x)\in C(\bar{\Omega})$, $0\le\varepsilon<\beta_1$. *Additionally, the following conditions are required to be satisfied:*

$$\varphi_0(0)=\tau(0),\quad b(0)\tau(\xi_0)+c(0)=\tau(a).$$

Then the integral equation

$$\varphi_1(t)+\int_0^t\varphi_1(\eta)\left[b(t)G_\xi(t,\xi_0,\eta,a)\right]d\eta=g_1(t),$$

has a unique regular solution given by

$$\varphi_1(t)=g_1(t)-\int_0^t g_1(\eta)R(t,\eta)d\eta, \tag{7}$$

where

$$g_1(t)=c(t)+b(t)\left[\int_0^t\varphi_0(\eta)G_\xi(t,\xi_0,\eta,0)d\eta+\int_0^a\tau(\xi)G(t,\xi_0,0,\xi)d\xi++\int_0^t\int_0^a f(\eta,\xi)G(t,\xi_0,\eta,\xi)d\xi d\eta\right]$$

and $R(t,\eta)$ *is resolvent of the kernel* $b(t)G_\xi(t,\xi_0,\eta,a)$.

Theorem 2. *If the conditions of the Lemma 1 are satisfied, then the solution of the Bitsadze-Samarskii problem (1)-(2)-(3)-(4) exists and is unique, and it is represented by (7) and (6).*

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